(b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

2.4. What can be said about the eigenvalues of a unitary matrix?

2.5. Let $S \in \mathbb{C}^{m \times m}$ be *skew-hermitian*, i.e., $S^* = -S$.

(a) Show by using Exercise 2.3 that the eigenvalues of S are pure imaginary.

(b) Show that $I - S$ is nonsingular.

(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the *Cayley transform* of S, is unitary. (This is a matrix analogue of a linear fractional transformation $(1+s)/(1-s)$, which maps the left half of the complex s-plane conformally onto the unit disk.)

2.6. If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is $\text{null}(A)$?

2.7. A *Hadamard matrix* is a matrix whose entries are all ± 1 and whose transpose is equal to its inverse times a constant factor. It is known that if A is a Hadamard matrix of dimension $m > 2$, then m is a multiple of 4. It is not known, however, whether there is a Hadamard matrix for every such m, though examples have been found for all cases $m \le 424$.

Show that the following recursive description provides a Hadamard matrix of each dimension $m = 2^k$, $k = 0, 1, 2, \ldots$:

$$H_0 = \begin{bmatrix} 1 \end{bmatrix}, \qquad H_{k+1} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix}.$$